Morin, Pat

**A fast algorithm for the product structure of planar graphs.** (English) Zbl 1522.05072

Algorithmica 83, No. 5, 1544-1558 (2021).

Given two graphs $G$ and $H$, their strong product $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$, and an edge between the vertices $(v, x)$ and $(w, y)$ if one of the following holds:

- $v = w$ and $xy \in E(H)$; or
- $x = y$ and $vw \in E(G)$; or
- $vw \in E(G)$ and $xy \in E(H)$.

For a graph $G$, its tree-decomposition is a collection $\{B_i : i \in I\}$ of subsets of $V(G)$, called bags, such that $\bigcup_{i \in I} B_i = V(G)$, along with a tree $T$ with $V(T) = I$, satisfying the following properties:

- $vw \in E(G) \implies v, w \in B_i$ for some $i \in I$; and
- for $i, j, k \in I$, if $j$ lies on the path of $T$ from $i$ to $k$, then $B_i \cap B_k \subseteq B_j$.

The width of a tree decomposition is one less than the size of the largest bag. The treewidth of $G$, denoted $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of $G$.

In [*V. Dujmović* et al., J. ACM 67, No. 4, Article No. 22, 38 p. (2020; Zbl 1466.05047)], the authors showed that every planar graph $G$ is a subgraph of $H \boxtimes P$, where $H$ has treewidth at most 8 and $P$ is a path. This result is used to resolve several open problems in graph theory on queue numbers, nonrepetitive chromatic numbers, labelling schemes, and $p$-centered colorings. Moreover, the proof produces a $O(n^2)$ time algorithm for finding $H$ and a mapping from $V(G)$ onto $V(H \boxtimes P)$.

The main result of this paper is to improve the run time to $O(n \log n)$, and this also naturally improves the results obtained as a consequence of the theorem of Dujmović et al. [loc. cit.]. The author raises the question of whether the run time can be improved further to $O(n)$. This has now been resolved in the positive by *P. Bose* et al. [An optimal algorithm for product structure in planar graphs. in: 18th Scandinavian symposium and workshops on algorithm theory, SWAT 2022, Tórshavn, Faroe Islands, June 27–29, 2022. Wadern: Schloss Dagstuhl – Leibniz-Zentrum für Informatik (2022; Zbl 1491.68015)]. It is also worth mentioning the work of *T. Ueckerdt* et al. [Electron. J. Comb. 29, No. 2, Research Paper P2.51, 12 p. (2022; Zbl 1527.05046)], which replaces "treewidth at most 8" in the result of Dujmović [loc. cit.] with "simple treewidth at most 6". To clarify, a tree-decomposition of a graph $G$ is $k$-simple if

- it has width at most $k$, and
- every set of $k$ vertices of $G$ is contained in at most two bags.

The simple treewidth of a graph $G$, denoted $\mathrm{stw}(G)$, is the minimum $k$ such that $G$ has a $k$-simple tree-decomposition.

Reviewer: Brahadeesh Sankarnarayanan (Mumbai)

**MSC:**

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|---|---|
| 05C10 | Planar graphs; geometric and topological aspects of graph theory |
| 05C75 | Structural characterization of families of graphs |
| 05C76 | Graph operations (line graphs, products, etc.) |
| 05C85 | Graph algorithms (graph-theoretic aspects) |
| 68Q25 | Analysis of algorithms and problem complexity |



**Keywords:**

planar graphs; product structure; queue number; nonrepetitive colouring; adjacency labelling; $p$-centered colouring

**Full Text:** DOI arXiv

**References:**

[1] Alon, N.; Grytczuk, J.; Hałuszczak, M.; Riordan, O., Nonrepetitive colorings of graphs, Random Struct. Algorithms, 21, 3-4, 336-346 (2002) · Zbl 1018.05032 · doi:10.1002/rsa.10057

[2] Dębski, M., Felsner, S., Micek, P., Schröder, F.: Improved bounds for centered colorings. In: Chawla, S. (ed.) Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020, pp. 2212-2226. SIAM (2020). doi:10.1137/1.9781611975994.136 · Zbl 07304160

[3] Diestel, R.: Graph Theory, Fifth Edition, volume 173 of Graduate texts in mathematics. Springer, Berlin (2017). doi:10.1007/978-3-662-53622-3 · Zbl 1375.05002

[4] Dujmović, V., Esperet, L., Joret, G., Walczak, B., Wood, D.R.: Planar graphs have bounded nonrepetitive chromatic number. In: Advances in Combinatorics, March (2020). doi:10.19086/aic.12100 · Zbl 1450.05012

[5] Dujmović, V., Joret, G., Micek, P., Morin, P., Ueckerdt, T., Wood, D.R.: Planar graphs have bounded queue-number. In: Proceedings of 60th Annual Symposium on Foundations of Computer Science (FOCS '19), (2019). arXiv:1904.04791 · Zbl 1466.05047

[6] Dujmović, V., Esperet, L., Gavoille, C., Joret, G., Micek, P., Morin, P.: Adjacency labelling for planar graphs (and beyond). CoRR, abs/2003.04280, (2020), arXiv:2003.04280

[7] Dujmović, V., Morin, P., Wood, D.R.: The structure of k-planar graphs. CoRR, abs/1907.05168, (2019), arXiv:1907.05168

[8] Gabow, HN; Tarjan, RE, A linear-time algorithm for a special case of disjoint set union, J. Comput. Syst. Sci., 30, 2, 209-221 (1985) · Zbl 0572.68058 · doi:10.1016/0022-0000(85)90014-5

[9] Grigoriev, A.; Bodlaender, HL, Algorithms for graphs embeddable with few crossings per edge, Algorithmica, 49, 1, 1-11 (2007) · Zbl 1131.68120 · doi:10.1007/s00453-007-0010-x

[10] Heath, LS; Leighton, FT; Rosenberg, AL, Comparing queues and stacks as mechanisms for laying out graphs, SIAM J. Discrete Math., 5, 3, 398-412 (1992) · Zbl 0764.05093 · doi:10.1137/0405031

[11] Kannan, S., Naor, M., Rudich, S.: Implicit representation of graphs. In: Janos, S. (ed.) Proceedings of the 20th Annual ACM Symposium on Theory of Computing, May 2-4, 1988, Chicago, Illinois, USA, pp. 334-343. ACM (1988) doi:10.1145/62212.62244 · Zbl 0768.05082

[12] Kannan, S.; Naor, M.; Rudich, S., Implicit representation of graphs, SIAM J. Discrete Math., 5, 4, 596-603 (1992) · Zbl 0768.05082 · doi:10.1137/0405049

[13] Korzhik, VP; Mohar, B., Minimal obstructions for 1-immersions and hardness of 1-planarity testing, J. Graph Theory, 72, 1, 30-71 (2013) · Zbl 1259.05046 · doi:10.1002/jgt.21630

[14] Král', D., Lazic, R.: Open problems from the workshop on algorithms, logic and structure, December (2016). https://warwick.ac.uk/fac/sci/maths/p

[15] Mohar, B., A linear time algorithm for embedding graphs in an arbitrary surface, SIAM J. Discrete Math., 12, 1, 6-26 (1999) · Zbl 0931.05025 · doi:10.1137/S089548019529248X

[16] Nešetřil, J.; de Mendez, PO, Tree-depth, subgraph coloring and homomorphism bounds, Eur. J. Comb., 27, 6, 1022-1041 (2006) · Zbl 1089.05025 · doi:10.1016/j.ejc.2005.01.010

[17] Nešetřil, J.; de Mendez, PO, Grad and classes with bounded expansion i. decompositions., Eur. J. Comb., 29, 3, 760-776 (2008) · Zbl 1156.05056 · doi:10.1016/j.ejc.2006.07.013

[18] Pilipczuk, M. Siebertz, S.: Polynomial bounds for centered colorings on proper minor-closed graph classes. In: Timothy M.C. (ed.) Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, pp. 1501-1520. SIAM, (2019). arXiv:1807.03683v2, doi:10.1137/1.9781611975482.91 · Zbl 1434.05056

[19] Urschel, J.C., Wellens, J.: Testing k-planarity is NP-complete. CoRR, abs/1907.02104, (2020). arXiv:1907.02104